An analytical solution for transient radial flow through unsaturated fractured porous media

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[1] This paper presents analytical solutions for one-dimensional radial transient flow through a horizontal, unsaturated fractured rock formation. In these solutions, unsaturated flow through fractured media is described by a linearized Richards’ equation, while fracture-matrix interaction is handled using the dual-continuum concept. Although linearizing Richards’ equation requires a specially correlated relationship between relative permeability and capillary pressure functions for both fractures and matrix, these specially formed relative permeability and capillary pressure functions are still physically meaningful. These analytical solutions can thus be used to describe the transient behavior of unsaturated flow in fractured media under the described model conditions. They can also be useful in verifying numerical simulation results, which as demonstrated in this paper, are otherwise difficult to validate.


1. Introduction

[2] In the past few decades, flow through unsaturated fractured rock, a special case of multiphase flow, has received a lot of attention because of subsurface environmental considerations. Quantitative analysis of flow in unsaturated fractured rock is often based on Richards’ equation. Because of its nonlinear nature, Richards’ equation solutions for general flow through fractured media may be obtained only with a numerical approach. On the other hand, analytical solutions, if available, provide more direct insight into the physics of unsaturated flow phenomena than do numerical or laboratory studies, and they are often needed to examine and verify numerical model schemes or results.

[3] For unsaturated flow through homogeneous single-porosity soils, many analytical solutions, both exact and approximate, have been developed, based on different levels of Richards’ equation linearization [e.g., Pullan, 1990; Warrick and Parkin, 1995; Basha, 1999; Philips, 1969; Zimmerman and Bodvarsson, 1995]. Despite the advances made so far, however, precise analytical solutions to Richards’ equation remain intractable under general flow conditions, because of its known nonlinearity. In addition, it becomes more difficult to obtain an analytical solution for flow through unsaturated fractured porous media because of the additional complexity introduced by fracture-matrix interaction.

[4] Recently, we presented a set of new analytical solutions for unsaturated flow within a single matrix block with fracture-matrix interaction [Wu and Pan, 2003]. These analytical solutions required a specially correlated relationship between relative permeability and capillary pressure functions. The present work extends our analytical solution approach to the entire fracture-matrix flow system, using a general dual-continuum approach. In this work, we show that it is possible to obtain analytical solutions if the specially correlated relative permeability and capillary pressure functions hold true for both fracture and matrix systems. In addition, we demonstrate that the new analytical solutions are very useful for checking numerical model results for unsaturated flow through fractured porous media.

2. Mathematical Formulation

[5] The problems to be solved are cases of unsaturated radial flow in a horizontal and uniform fracture-matrix formation corresponding to a fully penetrating injection well, with either constant well pressure or constant injection rate. The formation consists of identical cubic matrix blocks separated by a uniform three-dimensional fracture network, as in the Warren and Root model [Warren and Root, 1963]. In this work, the Warren and Root double-porosity model is extended into a general dual-continuum concept to include flow within matrix [e.g., Pruess and Narasimhan, 1985]. In the extended dual-continuum model, an “effective” porous continuum is adapted to approximate these two types of media (fractures and rock matrix), and unsaturated flow in fractured rocks is separately described using a doublet of Richards’ equations for the two continua. In particular, flow inside matrix blocks is handled fully transiently using a local spherical coordinate, which is different from the quasi-steady state method of the Warren and Root model. Furthermore, we assume that the two sets of capillary pressure and relative permeability functions for fracture and matrix, respectively, are in the form:

\[ k_{x_x}(S_x) = C_{k_x}(S_x) \]

and

\[ P_{c_x}(S_x) = P_{c_x} - P_{w_x} = C_{P_c}(S_x) \]

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where subscript $\xi$ is an index for fracture ($\xi = F$) or matrix ($\xi = M$); $P_{g\xi}$ is constant air (or gas) pressure in fractures or the matrix; $P_{w\xi}$ is liquid water pressure in fractures or the matrix, respectively; $C_{k\xi}$ and $C_{p\xi}$ (Pa) are coefficients, $\alpha_\xi$ and $\beta_\xi$ are exponential constants of relative permeability and capillary pressure functions, respectively, for fracture or matrix systems; and $S^*_\xi$ is the effective or normalized fracture or matrix water saturation,

$$
S^*_\xi = \frac{S_\xi - S_{tr}}{1 - S_{tr}}
$$

with $S_{tr}$ being the residual saturation value in fracture or matrix systems.

If the following condition:

$$
\alpha_\xi = \beta_\xi + 1
$$

is satisfied for both fractures and matrix, Richards’ equation can be readily linearized for flow through both fractures or the matrix [Wu and Pan, 2003]. In this work, however, the linearization of the continuity in capillary pressure on the matrix surface requires the exponential constants of capillary $\beta_\xi = 1$, leading to $\alpha_\xi = 2$, for both fracture and matrix media. The linearized governing equation for unsaturated radial flows (ignoring the gravity effect and the compressibility of water and rock) through the fractures can be derived by combining a mass balance on a control volume with the dual-continuum concept (see Appendix A), as follows:

$$
\frac{\partial^2 S_F}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial S_F}{\partial r} \right) - \frac{6\phi_M D_M}{B_{wp} D_p} \frac{\partial S_M}{\partial x} = \frac{1}{D_M} \frac{\partial S_F}{\partial t}
$$

where $\phi_M$ and $\phi_F$ are porosity for matrix or fractures (if not described, the symbols for variables and parameters in equation (5) or in the following equations are defined in the appendixes.) The third term on the left-hand side of equation (5) represents flow exchange terms on the local matrix interface between fracture and matrix systems, describing the continuity in mass flux.

[6] For flow inside the matrix, we use a 1-D spherical flow approximation, because the 1-D spherical flow within matrix blocks is the most commonly used in the literature for estimating matrix flow. In addition, the different shape and flow geometry of matrix blocks are found to have an insignificant effect on the fracture-matrix interaction of water and oil in fractured petroleum reservoirs [Wu and Pruess, 1988]. One-dimensional spherical unsaturated flow inside a cubic matrix block is then governed by [Wu and Pan, 2003]

$$
\frac{\partial^2 S_M}{\partial x^2} + \frac{2}{x} \frac{\partial S_M}{\partial x} = \frac{1}{D_M} \frac{\partial S_M}{\partial t}
$$

The initial conditions within fractures and matrix systems are uniform:

$$
S_\xi|_{t=0} = S_{tr}
$$

Note here that for simplicity, initial saturations in fractures and the matrix are set to their residual values.

[7] The first inner boundary condition is that the wellbore be specified with constant saturation:

$$
S_F(r = r_w, t) = S_0
$$

and the second is that the injection rate $q$ be:

$$
q = -\frac{2\pi r_w h_k x C_{kF} \partial S_F^*}{\mu_w} \left|_{r = r_w} \right. = q
$$

Far from the well, the saturation in the fractures remains at its initial value:

$$
S_F(r = \infty, t) = S_{F}\bar
$$

At the matrix surface, continuity in pressure or capillary pressure is enforced:

$$
P_m(r, t) = P_{mF}(x = B/2, t; r)
$$

while at the matrix block center, a zero-gradient condition is maintained for symmetry:

$$
\frac{\partial S_M(x = 0, t; r)}{\partial x} = 0
$$

The linearized equation system above is similar to the gradient flow model for single-phase flow through fractured reservoirs [Streltsova, 1983].

3. Analytical Solutions

[8] In the following dimensionless variables, the dimensionless distances and time are defined as

$$
r_D = \frac{r}{r_w}, \quad x_D = \frac{2x}{B} \quad \text{and} \quad t_D = \frac{D_m t}{(B/2)^2}
$$

The normalized (or scaled) water saturation is defined as the same as the effective saturation of equation (3).

[9] In terms of these dimensionless variables, the solution for the normalized matrix saturation in Laplace space is given by (Appendix B):

$$
\tilde{S}_{MD} = A_4 \frac{\tilde{S}_0}{\tilde{x_D}} \frac{I_{1/2}(\tilde{\alpha}_{MD})}{I_{1/2}(\tilde{\alpha})} = A_4 \frac{\tilde{S}_0}{\tilde{x_D}} \frac{\sinh(\tilde{\alpha}_{MD})}{\sinh(\tilde{\alpha})}
$$

where $\alpha = \sqrt{A_3 p}$ and $I_{1/2}$ is the modified Bessel function of the first kind.

[10] $\tilde{S}_{FD}$ is the solution of the normalized fracture saturation in Laplace space, defined differently for the two well boundary conditions (Appendix B). The solution $\tilde{S}_{FD}$ with constant water saturation at the wellbore, in equation (8), is given by

$$
\tilde{S}_{FD} = \frac{S_{FD}}{p} \frac{K_0(\sqrt{\tilde{x}_D})}{K_0(\sqrt{\tilde{x}})}
$$

where $x_2 = A_1 A_4 [\sigma \cdot \cosh(\sigma) - 1] + A_3 p$ and $A_4 = C_{pM}/C_{pF}$. For the case of constant flow rate as defined in equation (9), the solution $\tilde{S}_{FD}$ is

$$
\tilde{S}_{FD} = \frac{q_0}{p} \frac{K_0(\sqrt{\tilde{x}_D})}{\sqrt{x_2} K_1(\sqrt{x_2})}
$$
In the solutions above, \( K_0 \) and \( K_1 \) are the modified Bessel functions of the second kind of zero and first order, respectively.

Equations (14), (15), and (16) constitute the solutions in Laplace space for normalized fracture and matrix saturations under two types of inner well boundary conditions, which depend on four parameters \((A_1, A_2, A_3, \text{ and } A_4)\), relating to matrix size and ratios of porosity, absolute permeability, and relative permeability and capillary pressure coefficients for fracture and matrix systems, in addition to the dimensionless spatial and time variables. Analysis of these dimensionless variables and their interrelations indicates that the dimensionless solutions are characterized mainly by the ratios of permeability, storage, and capillary pressure coefficients of fractures and matrix, respectively. This shows that the analytical solutions are a little more complicated that the Warren and Root solution for single-phase flow, which is characteristic of only two parameters, \( \lambda \) and \( \omega \).

### 4. Application

To apply these solutions, we use the Stehfest [1970] algorithm to invert these solutions from Laplace space to real space. First, the analytical solutions of equations (15) and (16) were used to calculate type curves for transient flow through a fully penetrating well into a uniform, horizontal fractured formation, which is 10 m thick and radially infinite. The fractured formation consists of uniform, identical 1 \( \times \) 1 \( \times \) 1 m cubes of matrix blocks, surrounded by a uniform, 3-D fracture network, identical to the Warren and Root conceptual model. The basic fracture-matrix and fluid parameters used for the example are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Matrix</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \phi_M )</td>
<td>0.30</td>
<td>0.001</td>
</tr>
<tr>
<td>( k_F )</td>
<td>1.0 ( \times ) 10^{-15}</td>
<td>1.0 ( \times ) 10^{-12}</td>
</tr>
<tr>
<td>( S_F )</td>
<td>0.01 (Figures 1 and 2) or 0.2 (Figure 3)</td>
<td>0.1 (Figures 1 and 2) or 1.0 (Figure 3)</td>
</tr>
<tr>
<td>( \rho_M )</td>
<td>1.0</td>
<td>1.0 ( \times ) 10^{-4}</td>
</tr>
<tr>
<td>( \rho_F )</td>
<td>1.0 ( \times ) 10^{-3}</td>
<td>1.0 ( \times ) 10^{-3}</td>
</tr>
<tr>
<td>( \mu_M )</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>( \mu_F )</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>( r_w )</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>


![Figure 1. Type curves of normalized liquid saturation in fractures versus dimensionless radial distance at different dimensionless times, with \( S_F = 1 \) at well.](image)
the radial symmetric formation. The double-porosity grid represents the matrix system by one mesh locally [Warren and Root, 1963], equivalent to one-block approximation of the matrix continuum in the MINC approach [Wu and Pruess, 1988], while the MINC grid subgrids each matrix block with seven nested cells, for better numerical accuracy in estimating fracture-matrix flow [Pruess and Narasimhan, 1985].

Figure 3 shows the saturation distribution along the fractures in the radial direction at time of 0.1 and 10 days, respectively, simulated by the analytical, double-porosity, and MINC modeling results. Note that the physical process simulated in this example is extremely nonlinear and dynamic. The initial liquid saturations are at residual values for both fracture and matrix systems. At the beginning, the boundary saturation for fractures at the wellbore jumps to one (thus flow rate at the well becomes infinitely large). Once imbibed into the fractures, the liquid will be competed by two forces in two directions, one for continuous flow along fractures away from the well, and the other sucked into dry matrix blocks.

The numerical model results, shown in Figure 3, indicate that the MINC model does a much better job in matching the analytical solutions than the double-porosity model. This implies that in this case, the MINC concept better captures these physical processes by considering capillary gradients at the matrix surface and inside matrix blocks. However, Figure 3 shows that even the MINC simulations with seven cells cannot match the analytical results very well, because the matrix surface is subject to dynamic boundary conditions (i.e., varying fracture saturation or capillary forces, which occurs at the upstream boundary for initializing imbibition into the matrix). Using a dual-continuum numerical approach [Wu and Pan, 2003], extremely refined spatial discretization (30 MINC cells) is required to model imbibing processes accurately under dynamic upstream boundary conditions.

Although the analytical solutions presented above are obtained under very strict assumptions (i.e., specially correlated relations between relative permeability and capillary functions for both fractures and the matrix, as well as negligible gravity effects), the relative permeability and capillary functions of equations (1) and (2) are not only physically meaningful, but also among the most widely used relations [Honarpour et al., 1986]. Since the treatment of fracture-matrix interaction in this analytical model is based rigorously on the dual-continuum concept, the analytical solutions obtained can be useful in evaluating numerical model results from dual-continuum models, as shown above.

5. Concluding Remarks

This paper shows that with specially formed capillary pressure and relative permeability functions, it is possible to obtain analytical solutions for transient unsaturated flow in fracture-matrix systems using the commonly used dual-continuum concept. With the analytical solutions provided in Laplace space, analytical solutions in real space can be readily obtained using numerical inversion techniques. The analytical solution approach of this work can be readily extended to other boundary conditions or different flow geometries, such as linear and multidimensional unsaturated flow through fractured porous formations.

The analytical solutions, though limited by the assumptions for their applications, can be used to obtain some insight into the physics of transient flow processes related to fracture-matrix interactions. As demonstrated in this work, these analytical solutions are very useful in verifying numerical models and their results for describing flow through unsaturated fractured rock, especially the flow
through a fracture-matrix interface, which is otherwise very difficult to evaluate.

Appendix A: Derivation of Governing Equations

[21] Let us consider the situation of unsaturated flow in a horizontal and uniform fracture-matrix formation using the Warren and Root conceptual model for fracture network and matrix blocks. Furthermore, gravity effects are ignored and incompressible liquid flows through a single well into a radial infinite system. The governing equations of unsaturated radial flow through such a fracture-matrix system can be derived by combining a mass balance on the control volume with the dual-continuum concept [Lai et al., 1983]. In the radial system, a control (bulk) volume at a radial distance of r from the well is defined as

\[ V_n = \pi \left( (r + dr)^2 - r^2 \right) h \approx 2\pi rh dr \]  \hspace{1cm} (A1)

where h is the thickness of formation.

[22] The interface area \( A_c \) between rock matrix blocks and surrounding fractures within the control volume, when the volume fraction of fractures can be ignored, \( V_n \) is written as

\[ A_c = 6B^2 V_n \]  \hspace{1cm} (A2)

where B is the dimension of matrix cubes.

[23] Mass balance for the incompressible fluid for the fracture system within the control volume requires that:

\[ q_r A_c - \left[ q_r A_c + \frac{\partial}{\partial r} (q_r A_c) dr \right] + [q_r A_c]_{r-B/2} = \frac{\partial (\rho \phi_w S_F)}{\partial r} \]  \hspace{1cm} (A3)

where x is the distance from a nested cross-sectional surface within the matrix block (having an equal distance to the matrix surface) to the center of the cube (i.e., a one-dimensional spherical coordinate with its center within the matrix block), and \( q_r \) and \( q_x \) are Darcy’s velocity along r and x directions, respectively, calculated as

\[ q_r = -\frac{k_F C_F}{\mu_w} \frac{\partial P_{SF}}{\partial r} = \frac{k_F C_F}{\mu_w} \frac{\partial P_{SF}}{\partial r} \]  \hspace{1cm} (A4)

and

\[ q_x = -\frac{k_M k_M}{\mu_w} \frac{\partial P_{SM}}{\partial x} = \frac{k_M k_M}{\mu_w} \frac{\partial P_{SM}}{\partial x} \]  \hspace{1cm} (A5)

where \( \mu_w \) is dynamic water viscosity. Substituting (A4) and (A5), as well as, radial cross area \( A_r = 2\pi rh \) into (A3), yields

\[ \frac{k_F C_F}{\mu_w} (1 - S_{Fr}) \left( \frac{\partial S_{FD}}{\partial r} \right) = \frac{k_M k_M}{\mu_w} \frac{\partial P_{SM}}{\partial x} \]  \hspace{1cm} (A6)

If we define \( D_f \) as soil water or moisture diffusivity [Wu and Pan, 2003],

\[ D_f = \frac{k_F C_F}{\phi_w \mu_w} \frac{\partial S_{FD}}{\partial x} = \frac{k_M C_M}{\phi_w \mu_w (1 - S_{Fr})} \]  \hspace{1cm} (A7)

with a dimension of \( m^2/s \). We thus derive the linearized flow governing equation (5) for flow through the fractures.

Appendix B: Derivation of Analytical Solutions

[24] In terms of the dimensionless variables (equations (3) and (13)), the two governing equations become:

\[ \frac{\partial^2 S_{FD}}{\partial r^2} + \frac{1}{r} \frac{\partial S_{FD}}{\partial r} - A_1 \frac{\partial S_{MD}}{\partial x} \bigg|_{x=1} = A_2 \frac{\partial S_{FD}}{\partial r} \]  \hspace{1cm} (B1)

and

\[ \frac{\partial^2 S_{MD}}{\partial x^2} + \frac{2}{x} \frac{\partial S_{MD}}{\partial x} = A_3 \frac{\partial S_{MD}}{\partial D} \]  \hspace{1cm} (B2)

where

\[ A_1 = \frac{12 D_f \phi_w C_F}{D_f \phi_w C_F} \left( 1 - \frac{1}{S_{Fr}} \right), \hspace{1cm} A_2 = \frac{4 r^2}{B^2}, \hspace{1cm} \text{and} \hspace{1cm} A_3 = \frac{D_F}{D_M} \]  \hspace{1cm} (B3)

The initial condition becomes

\[ S_{FD} |_{t_0=0} = 0 \]  \hspace{1cm} (B4)

The boundary condition of constant saturation at the well becomes

\[ S_{FD} (r_D = 1, t_D) = \frac{S_0 - S_{Fr}}{1 - S_{Fr}} = S_{ID} \]  \hspace{1cm} (B5)

The constant rate condition turns into

\[ \frac{\partial S_{FD}}{\partial D} \bigg|_{t_D=1} = \frac{q_D}{2\pi h k_F C_F C_{PF}} = q_D \]  \hspace{1cm} (B6)

Far from the well,

\[ S_{FD} (r_D = \infty, t_D) = 0 \]  \hspace{1cm} (B7)

At the matrix surface, the continuity in pressure becomes

\[ S_{MD} (x_D = 1, t_D; t_D) = \frac{C_M}{C_{PF}} S_{FD} (r_D = r_D, t_D) \]  \hspace{1cm} (B8)

Note that this linear relationship for the continuity in pressure on the matrix surface assumes the exponential constants of capillary \( \beta_c = 1 \) for both fracture and matrix media. At the matrix block center,

\[ \frac{\partial S_{MD}}{\partial x} \bigg|_{x_D=0} = 0 \]  \hspace{1cm} (B9)

Applying the Laplace transformation to equations (B1) and (B2) and incorporating the initial condition (B4) yield

\[ \frac{\partial^2 S_{FD}}{\partial r^2} + \frac{1}{r} \frac{\partial S_{FD}}{\partial r} - A_1 \frac{\partial S_{MD}}{\partial x} \bigg|_{x=1} - pA_2 S_{FD} = 0 \]  \hspace{1cm} (B10)

and

\[ \frac{\partial^2 S_{MD}}{\partial x^2} + \frac{2}{x} \frac{\partial S_{MD}}{\partial x} = pA_3 S_{MD} = 0 \]  \hspace{1cm} (B11)
where $\bar{S}_{2D}$ is the Laplace transformed normalized saturation, and $p$ is the Laplace variable. Then the solutions of normalized matrix and fracture saturations, subject to the Laplace transformed boundary conditions from equations (B5)–(B9), are given by equations (14), (15) and (16).

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