An Analytical Solution for Wellbore Heat Transmission in Layered Formations

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Summary. This paper presents a new analytical solution for wellbore heat transmission. Previous treatments of the wellbore heat-transfer problem are improved in several aspects: (1) nonhomogeneous formations are approximated as layered formations with different physical properties; (2) closed-form analytical solutions are obtained in both real and Laplace space; and (3) a more accurate formula is provided for the transient heat-conduction function, \( f(t_p) \).

Introduction

Heat is transferred to or from the wellbore when there is a difference in temperature between the surrounding formation and the injected (or produced) fluid. To evaluate the feasibility of a thermal recovery project, it is necessary to estimate the heat losses or gains of the flowing fluid in wells, the changes in temperature with time and depth, and the heat-transfer conditions between wellbore and formation. A quantitative description of heat exchange between a wellbore and surrounding formations often is also required when one attempts to estimate formation temperatures from wellbore measurements.

Studies of wellbore heat transmission during hot or cold fluid injection have appeared in the literature since the 1950's. The techniques available for dealing with wellbore heat transmission include analytical and numerical methods. Lessem \textit{et al.} \(^1\) and Squire \textit{et al.} \(^2\) derived and solved similar systems of differential equations describing the temperature behavior of gas and hot-water injection wells. They neglected wellbore thermal resistance and made the following assumptions.

1. No conductive heat transfer occurs in the vertical direction of either the flowing fluid or the formation.
2. The mass flow rate of gas or water is constant throughout the injection or production system.
3. The volumetric heat capacities of fluids and formation are constant.
4. The formation is homogeneous and isotropic with constant thermal conductivity.
5. The fluid temperature is the same as the formation temperature on the wellbore surface.

Subsequent work introduced another assumption, that vertical heat transfer in the wellbore was considered steady state.

The classic study by Ramey \(^3\) on wellbore heat transmission improved Moss and White's approach to incorporate an overall heat-transfer coefficient. Ramey presented an approximate solution for the temperatures of fluids, tubing, and casing as a function of time and depth in a well used for hot-fluid injection. Satter \(^4\) suggested a similar method for analyzing wellbore heat loss when condensing steam flow is considered and provided a sample procedure for a given set of reservoir properties. Ramey \(^5\) and Willhite \(^6\) gave an expression for the overall heat-transfer coefficient for any well completion and the early-time values of the transient heat-conduction function. Durrant and Thambayanag's \(^7\) more recent work provided an iterative procedure for the wellbore heat-transmission problem during flow of steam/water mixtures that includes vertical heat conduction.

The numerical models by Farouq Ali \(^8\) and Wooley \(^9\) were more comprehensive than the analytical models. They include both horizontal and vertical heat conduction in the formation and can deal with different well operation conditions. The numerical methods, however, are often too complicated for field applications or for reservoir simulation studies because many of the required wellbore and formation heat-transfer properties are rarely known precisely.

The mathematical model for wellbore heat transmission presented in this paper adopts assumptions similar to those of Lessem \textit{et al.} \(^1\). The main differences are that we introduce an overall heat-transfer coefficient to consider the wellbore heat resistance and that we treat the surrounding earth as consisting of an arbitrary number of layers with different thermal and physical properties and arbitrary initial temperature distributions (Fig. 1). An analytical solution has been obtained in both real and Laplace space for prediction of wellbore heat transmission. The numerical results calculated from the analytical solutions are compared with Ramey's long-time approximation. Illustrative applications are given for predicting wellbore heat transmission for engineering designs of reservoir simulation studies in petroleum and geothermal reservoir development.

Mathematical Model

The transient heat-transmission problem under consideration is illustrated in Fig. 1. The injection (or production) well is cased to the top of the injection (or production) interval. Heat is transferred along the wellbore solely by convection and then by conduction into the formation. The formation consists of \( n \) layers with different thermal and physical properties. The system is composed of three parts, as Fig. 1 shows: (1) fluid-flow conduit inside the tubing; (2) tubing/casing annulus, casing wall, and cement; and (3) infinite formation surrounding the casing. The major assumptions and approximations are as follows.

1. Fluid flow in the tubing is 1D, vertical, and steady with constant mass flow rate.
2. The well fluid temperature is lumped radially.
3. The vertical heat conduction is neglected compared with heat convection by the flowing fluid.
4. Radial heat flow between the wellbore and the formation is steady state.
5. In the surrounding earth, the initial geothermal gradient is a known function of depth.
6. The vertical heat conduction in the formation can be ignored compared with the horizontal heat flow.

All other assumptions are similar to those of previous work. Therefore, the heat-transfer equation in the tubing can be written as

\[
\rho \cdot c_p \cdot (\partial T_{ij} / \partial t) + (2/r_i) q_f^a + \rho \cdot c_p \cdot v_m (\partial T_{ij} / \partial z) = 0,
\]

\( j = 1,2, \ldots , n \) \( \quad (z_{j-1} < z < z_j) \) \hspace{1cm} (1)

for liquid flow and

\[
\rho \cdot c_p \cdot (\partial T_{ij} / \partial t) + (2/r_i) q_f^a + \rho \cdot c_p \cdot v_m (\partial T_{ij} / \partial z) \pm (g/c_p) = 0,
\]

\( j = 1,2, \ldots , n \) \( \quad (z_{j-1} < z < z_j) \) \hspace{1cm} (2)

for gas flow, where \( g \) is the plus sign on the potential-energy term is used for flow down the well and the negative sign is used for flow up the well.\(^3\)

The heat conduction in Layer \( j \) of the formation is described by

\[
(1/r)(\partial^2 T_{ij} / \partial r^2) + (k_{ij} \cdot (\partial T_{ij} / \partial r)) = 0,
\]

\( j = 1,2, \ldots , n \) \( \quad (z_{j-1} < z < z_j) \) \hspace{1cm} (3)
The heat flux across the tubing surface is
\[ q_j^* = U_j (T_{1j} - T_{2j}) \big|_{r_1 = r_w}, \] (4)
where \( U_j \) = overall heat-transfer coefficient defined by Ramey and Willhite.\(^6\)

The initial condition in the well is
\[ T_{1j}(z, t=0) = T_{0j}(z) (\text{known functions}), j = 1, 2, \ldots n \ (z_{j-1} \leq z \leq z_j), \] (5)
and initial conditions in the formation are
\[ T_{2j}(r, z, t=0) = T_{0j}(z), \ j = 1, 2, \ldots n \ (z_{j-1} \leq z \leq z_j). \] (6)
In Eqs. 5 and 6 it is required that
\[ T_{1j}(z=0, t) = T_{air}, \] (7)
i.e., the temperature on the ground is assumed constant.
The boundary conditions are
\[ T_{1j}(z = 0, t) = T_{air}, \] (8)
and
\[ \lim_{r \to 0} T_{2j}(r, z, t) = T_{0j}(z), \ j = 1, 2, \ldots n. \] (9)

**Analytical Solution**
The dimensionless parameters for radial distance, time, and depth are
\[ r_D = r/r_w, \] (10)
\[ t_D = t/m_D, \] (11)
and \( z_D = z/D. \) (12)

The dimensionless temperature functions are
\[ \theta_j(z_D, t_D) = (T_{1j}(z, t) - T_{0j}(z))/(T_{air} - T_{0j}), \ \ j = 1, 2, \ldots n, \] (13)
in the wellbore and
\[ \phi_j(r_D, z_D, t_D) = (T_{2j}(r, z, t) - T_{0j}(z))/(T_{air} - T_{0j}), \ \ j = 1, 2, \ldots n, \] (14)
in the formation.
The unsteady-state solution of this system in Laplace space becomes (Appendix A)
\[ \theta_j(z_D, s_D) = C_j(s_D) \exp \left\{ \left[ -s + \beta_j - D_j(s_D) \right] s_D \right\} \]
\[ + Y_j(z_D, s), \ j = 1, 2, \ldots n. \] (15)
where \( C_j(s) = \left( 1/s \right) + \left( \delta_j(z_D) \right) / s \left( s + \beta_j - D_j(s) \right), \) \( j = 1, 2, \ldots n. \) (16)

\[ C_j(s) = \frac{\delta_j(z_D)}{s(s + \beta_j - D_j(s))} \] \times \exp \left\{ \left[ s + \beta_j - D_j(s) \right] z_D \right\}, \ j = 2, 3, \ldots n, \] (17)
and \( \delta_j(z_D, s_D) = \left[ \left( \delta_j(z_D) \right) K_1(\sqrt{s} z_D) \right] \left[ \left( \delta_j(z_D) \right) K_1(\sqrt{s} z_D) \right], \ j = 1, 2, \ldots n. \) (18)

The functions \( Y_j(z_D, s), D_j(s), \) and \( \delta_j(z_D) \) are defined in Appendix A. The temperature function, \( \theta_j, \) in Laplace space can be determined recursively from Layer \( j \) to Layer \( j+1 \) \( (j = 1, 2, \ldots n-1) \) because it is assumed that there is no vertical heat conduction in either the wellbore or the formation. Therefore, downstream wellbore fluid or formation temperatures have no effect on upstream temperatures.

Another important variable of interest for wellbores is the heat flow rate transferred into (or from) the formation. For a linear initial temperature distribution in each layer of the formation,
\[ T_j(z) = T_{cj} + g_{kj} z, \] (19)
where the \( T_{cj} \) are constant. Continuity at the interfaces of layers requires that
\[ T_{cj} = T_{air}, \] (20)
and \( T_{cj} + g_{kj} z_j = T_{cj+1} + g_{kj+1} z_j, j = 1, 2, \ldots n-1. \) (21)

Then \( T_j(z) = g_{kj} z \) \( (j = 1, 2, \ldots n) \) and
\[ Y_j(z_D, s) = \left[ \left( \delta_j(s + \beta_j - D_j(s)) \right) \right] L_j. \] (22)

For the heat flux into (or from) the formation, in Laplace space,
\[ Q_j(z_D, s) = -U_j \left( T_{air} - T_{0j} \right) \left( \delta_j(z_D, s) \right) \left( \delta_j(z_D, s) \right). \] (23)
(24)

The cumulative heat flow rate is
\[ Q(z_D, s) = \frac{2 \pi r_D D^2 (T_{air} - T_{0j})}{v_m} \sum_{j=1}^{n} \frac{U_j}{s} \left[ 1 - \frac{D_j(s)}{\beta_j} \right] \]
\[ \times \left\{ \left[ \delta_j(z_D-1, s) \right] + \frac{\delta_j(z_D, s)}{s(s + \beta_j - D_j(s))} \right\} \]
\[ \times \left( 1 - \exp \left\{ \left[ s + \beta_j - D_j(s) \right] (z_D - z_D-1) \right\} \right) \]
\[ + \frac{\delta_j(z_D, s)}{s(s + \beta_j - D_j(s))} \left( z_D - z_D-1 \right) \] (25)

The solutions in Laplace space can be evaluated by numerical inversion techniques.\(^9\) Analytical solutions in real space are desirable for validating the numerical inversion results and for predicting the early-time transient behavior of the system because the numerical Laplace transform cannot be expected to give accurate results for early time. We have obtained solutions in real space for a linear initial temperature distribution in each formation layer in

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TABLE 1—CALCULATION DATA

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_p$, °C/m [°F/ft]</td>
<td>0.03 [0.016]</td>
</tr>
<tr>
<td>$T_m$, °C [°F]</td>
<td>100 [212]</td>
</tr>
<tr>
<td>$Q$, m³/°C [ft³/D]</td>
<td>100 [3,531]</td>
</tr>
<tr>
<td>$\rho_w$, kg/m³ [lbm/ft³]</td>
<td>958 [59.8]</td>
</tr>
<tr>
<td>$U$, W/m²·°C [Btu/(ft²·°F)]</td>
<td>4196 [1.00]</td>
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Sandstone

<table>
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</thead>
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<td>$\rho_s$, kg/m³ [lbm/ft³]</td>
<td>2200 [137.3]</td>
</tr>
<tr>
<td>$c_s$, J/(kg·°C) [Btu/(lbm·°F)]</td>
<td>740 [0.167]</td>
</tr>
<tr>
<td>$K_s$, W/m·°C [Btu/(ft-hr·°F)]</td>
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</tbody>
</table>

Clay

<table>
<thead>
<tr>
<th>Property</th>
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</thead>
<tbody>
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<td>$\rho_c$, kg/m³ [lbm/ft³]</td>
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</tr>
<tr>
<td>$c_c$, J/(kg·°C) [Btu/(lbm·°F)]</td>
<td>800 [0.191]</td>
</tr>
<tr>
<td>$K_c$, W/m·°C [Btu/(ft-hr·°F)]</td>
<td>1.4 [0.81]</td>
</tr>
</tbody>
</table>

*16.06 cm [6.33 in.] ID.

Appendix B. For Layer 1 (0 ≤ z_D ≤ z_{Dj}) or for a homogeneous formation we have

$$ \theta_1(z_D, t_D) = \begin{cases} I_1 & (t_D \leq z_D) \\ I_1 + I_2 + I_3 & (t_D > z_D) \end{cases} \quad \quad \quad (26) $$

where

$$ I_1 = \frac{4\delta}{\pi^2} \int_0^{\infty} \frac{D^*_1(u)}{u} \left[ \exp\left( -\frac{u^2}{\sigma_1} t_D \right) - 1 \right] \frac{1}{u} \left[ D_1^*(u) \left( \beta_1 - \frac{u^2}{\sigma_1} \right) - R_1(u) \right]^2 + \frac{4}{\pi^2} \right] \, du, \quad \cdots \cdots (27) $$

$$ I_2 = \frac{2e^{-\beta_1 z_D}}{\sigma_1} \int_0^{\infty} \left[ 1 - \exp\left( -\frac{u^2}{\sigma_1} (t_D - z_D) \right) \right] \sin \left( \frac{2z_D}{\pi D_1^*(u)} \right) \, du, \quad \cdots \cdots (28) $$

$$ I_3 = \frac{2\delta_1 e^{-\beta_1 z_D}}{\pi} \int_0^{\infty} \frac{D_1^*(u)}{u} \left[ \left( \beta_1 - \frac{u^2}{\sigma_1} \right) - R_1(u) \right]^2 + \frac{4}{\pi^2} \right] \, du, \quad \cdots \cdots (29) $$

For Layer j = 2, 3, ..., n, the dimensionless wellbore temperatures are

$$ \theta_j(z_D, t_D) = A_j(t_D) + \int_0^{t_D} \left[ \theta_{j-1}(z_{Dj-1}, \tau) - A_j(\tau) \right] B_j(z_D, t_D - \tau) \, d\tau, \quad \cdots \cdots (30) $$

where

$$ A_j(t_D) = \frac{4\delta_j}{\pi^2} \int_0^{\infty} \frac{D_j^*(u)}{u} \left[ \exp\left( -\frac{u^2}{\sigma_j} t_D \right) - 1 \right] \frac{1}{u} \left[ D_j^*(u) \left( \beta_j - \frac{u^2}{\sigma_j} \right) - R_j(u) \right]^2 + \frac{4}{\pi^2} \right] \, du, \cdots \cdots (31) $$

and

$$ B_j(z_D, t_D) = \left\{ \begin{array}{ll}
0 & (t_D \leq z_D - z_{Dj-1}) \\
\frac{2}{\pi \sigma_j} \exp\left[ -\beta_j (z_D - z_{Dj-1}) \right] \int_0^{\infty} \sin \left( \frac{2z_D - z_{Dj-1}}{\pi D_j^*(u)} \right) \, du & (t_D > z_D - z_{Dj-1}) \end{array} \right. \cdots \cdots (32) $$

$$ R_j(u) \quad \text{and} \quad D_j^*(u) \quad \text{in Eqs. 27 through 32 are defined in Appendix B}.$$
Discussion

A series of tests were conducted to validate the analytical solutions. The numerical inversion results of the Laplace-transformed solution of Eq. 15 were compared with the numerical integration of the exact solution of Eq. 26 and with Ramey's long-time solution. The integrals in Eq. 26 were calculated with the numerical integral-evaluation routine from the NAG Fortran Library on a CRAY computer. Convergence of the numerical integration was very rapid and smooth.

The example problem is a hot-water injection at a constant rate. Table 1 gives the fluid and formation data for the calculation. As Fig. 2 shows, the numerical Laplace inversion results are in excellent agreement with the exact solution, and at long times, the solution and Ramey's solution converge to the same curve.

The results from the numerical Laplace inversion by the Stehfest algorithm generally need to be checked against some other solution, particularly for early times. We have found that the numerical inversion gives very poor results for the early times when \( t_D \ll z_D \). This probably occurs because of the rapidly changing condition at the sandface until the entire wellbore is full of injected water when \( t_D > 1 \). At later times, the numerical inversion will give very accurate results. Therefore, the exact solution in real space, instead of that in Laplace space, should be used for applications in which early-time transient behavior is important, such as in temperature well logging analysis.

As in most studies on wellbore heat transfer, the vertical heat conduction is ignored here because it is negligible compared with horizontal flow. We examine this approximation by comparing the horizontal and vertical temperature gradients in the formation derived from the solutions obtained above. As Fig. 3 shows, the ratio of vertical and horizontal temperature gradients is always smaller than 1%, and reaches its maximum around the temperature penetration fronts. A larger vertical heat flow may occur on the interface of formation layers with different properties where the temperatures obtained by neglecting vertical flow are vertically discontinuous.

We found that the difference in temperatures is very small on the interface of the sandstone and clay whose properties are given in Table 1. These results should be conservative because vertical temperature differences are overestimated if vertical flow is neglected. Therefore, the assumption that the vertical heat flow in the formation is negligible is probably acceptable for most engineering calculations.

A steady-state approximation for vertical heat transfer in the wellbore has been made in almost all previous wellbore heat-transfer models. This approximation is not needed here. It has been shown that the steady-state treatment overestimates the temperature increase in the wellbore at early times but that the differences disappear at long times.

The transient heat-conduction function, \( f(t_D) \), discussed in detail by Ramey and Willhite, is widely used for wellbore heat-transfer calculations. However, only an approximate analytical expression for \( f(t_D) \), which is based on the long-time line-source solution from Ramey, is available in the literature. We obtain an accurate formula for \( f(t_D) \) as a special case of applications of the analytical solution to a uniform and homogeneous formation (subscripts omitted):

\[
f(t_D) = \frac{2\pi K(T_2)_{r=r_w} - T_1(z)}}{\frac{dQ}{dz}} \frac{\phi(1,z_D,t_D)}{\omega(\theta(z_D,t_D) - \theta(1,z_D,t_D))}.
\]

It is interesting to note that in a more rigorous formulation, \( f(t_D) \) is a function not only of dimensionless time, \( t_D \), but also of dimensionless depth, \( z_D \). This can be seen explicitly from Fig. 4, in
which $f(t_D)$ from Eq. 35 is plotted for different $z_D$. Only after
$t_D \geq 500$ (2,500 hours for this case) does $f(t_D)$ become indepen-
dent of $z_D$. This means that the use of an $f(t_D)$ independent of $z_D$
will not give accurate results during the early transient time for well-
bore heat-transfer problems.

Heat loss (or gain) from wells is important for evaluating a thermal
recovery project. Figs. 5 and 6 show the behavior of heat flux and
cumulative heat transfer into the surrounding formation for hot-water
injection into a well in a homogeneous sandstone formation. Table 1
gives the calculation parameters. Fig. 6 clearly shows that the heat
losses from the well and the temperatures never reach a steady state
because the formation is modeled as an infinite radial system.

In an actual reservoir, formations are neither uniform nor
homogeneous, and layered formations may be a realistic approxima-
tion. To take into account effects of formation heterogeneity on well-
bore heat transfer, the temperature distribution along the wellbore
was calculated for hot-water injection into a formation consisting of
two layers. The upper 500 m [1,640 ft] is sandstone, and the lower
500 m [1,640 ft] is clay. Table 1 gives the problem parameters.
As Fig. 7 shows, if only sandstone properties are used, well
temperatures are underestimated because thermal diffusivity in sand-
stone is larger than that in clay. Fig. 7 suggests that the assumption
of constant formation properties introduces errors for nonhomo-
genous reservoirs.

**Conclusions**

An analytical solution for determining wellbore heat transfer has
been developed that is applicable to field predictions and reservoir
simulation studies of wellbore heat transmission in uniform and
layered formations. Illustrative examples were given for temperature
distributions along the wellbore and in the formation, and for heat-
transfer rates and cumulative heat loss (or gain) between wellbore
and formation. Analysis of the calculated results of the analytical
solution leads to the following conclusions.

1. Vertical heat conduction in the formation may be ignored for
engineering applications.

2. The use of a depth-independent heat-conduction function,
$f(t_D)$, will introduce large errors at early times in calculations of
wellbore temperature.

3. Effects of formation heterogeneity should be included for more
accurate predictions of wellbore heat transmission in nonhomo-
genous formations.

**Nomenclature**

- $A_j(t_D) = \text{defined in Eq. 31}$
- $A_j(s) = \text{defined in Eq. B-2}$
- $B_j(z_D,t_D) = \text{defined in Eq. 32}$
- $\bar{B}_j(z_D,s) = \text{defined in Eq. B-3}$
- $c_f = \text{formation specific heat of Layer } j, \text{J/kg·°C}$
- $c_f = \text{fluid specific heat in tubing, J/(kg·°C)}$
- $C_f = \text{constant in Eqs. A-23 and A-24}$
- $D = \text{depth to top of permeable interval, m [ft]}$
- $D_i(s) = \text{defined in Eq. A-25}$
- $f(t_D) = \text{transient heat-conduction function defined in}$
- $\text{Eq. 35}$
- $f_j(t_D) = \text{defined in Eq. B-7}$
- $f_j(s) = \text{defined in Eq. B-2}$
- $g = \text{acceleration of gravity, m/s}^2 [\text{ft/sec}^2]$ 
- $g_f = \text{geothermal gradient of Layer } j, ^{°C/m}$
- $g_j(z_D,t_D) = \text{defined in Eq. B-12}$
- $g_j(r_D,t_D) = \text{defined in Eq. B-6}$
- $i = \text{injection rate, m}^3/\text{s [B/D]}$
- $I_j = \text{defined in Eqs. 27 through 29 (j=1,2,3)}$
- $J_0 = \text{zero-order Bessel function of first kind}$
- $J_1 = \text{first-order Bessel function of first kind}$
- $k_h = \text{thermal conductivity of formation, W/(m·°C)}$
- $[\text{Btu/(hr·ft}^2·°\text{F/in.}]]$
\[ k_{hf} = \text{thermal conductivity of Layer } j, \text{ formation, W/(m} \cdot \text{°C) [Btu/(hr} \cdot \text{ft}^2 \cdot \text{°F}/\text{in.}]}. \]

\[ K_0 = \text{zero-order modified Bessel function of second kind}. \]

\[ K_1 = \text{first-order modified Bessel function of second kind}. \]

\[ n = \text{total formation layer number with different physical properties}. \]

\[ q'' = \text{heat flux between tubing and sandface, W/m}^2 \text{ [Btu/hr} \cdot \text{ft} \cdot \text{hr}]. \]

\[ \tilde{q}^j (r, s) = \text{defined in Eq. 24}. \]

\[ \tilde{Q}_c = \text{cumulative heat exchange between well and formation, J [Btu]}. \]

\[ \tilde{Q}_c (r) = \text{defined in Eq. 25}. \]

\[ r = \text{radius, m [ft]}. \]

\[ r_D = \text{dimensionless radius defined in Eq. 10}. \]

\[ r_i = \text{inside radius of tubing, m [ft]}. \]

\[ r_w = \text{outside radius of cement zone, m [ft]}. \]

\[ R_j (u) = \text{defined in Eq. B-14}. \]

\[ s = \text{Laplace transform variable}. \]

\[ t = \text{time, seconds}. \]

\[ t_D = \text{dimensionless time defined in Eq. 11}. \]

\[ T = \text{temperature, °C [°F]}. \]

\[ T_{air} = \text{surface temperature, °C [°F]}. \]

\[ T_f = \text{constant temperature in } T_j (z), \text{ °C [°F]}. \]

\[ T_j (z) = \text{initial temperature as a function of depth, °C [°F]}. \]

\[ T_{inj} = \text{surface injection fluid temperature, °C [°F]}. \]

\[ T_{1j} (r, z) = \text{temperature along tubing, °C [°F]}. \]

\[ T_{2j} (r, z) = \text{temperature in formation, °C [°F]}. \]

\[ f_i = \text{overall heat-transfer coefficient, W/(m}^2 \cdot \text{°C) [Btu/(hr} \cdot \text{ft}^2 \cdot \text{°F}]}. \]

\[ v_m = \text{mean flow speed inside tubing, m/s [ft/sec]}. \]

\[ Y_j (z, D) = \text{particular solution in Eq. 21}. \]

\[ Y_0 = \text{zero-order Bessel function of second kind}. \]

\[ Y_1 = \text{first-order Bessel function of second kind}. \]

\[ z = \text{vertical coordinate, m [ft]}. \]

\[ z_D = \text{dimensionless vertical coordinate defined in Eq. 12}. \]

\[ z_{Dj} = z_j / R_j, \text{ dimensionless depth of Layer } j, \text{ (j=1,2...n)}. \]

\[ z_j = \text{depth of bottom of Layer } j, \text{ (j=1,2...n, } z_0 = 0). \]

\[ \alpha = \text{thermal diffusivity of formation (α=K/ρc), m}^2 / \text{sec}. \]

\[ \beta_j = \text{dimensional constant (Eq. A-3)}. \]

\[ \beta_j (D, r_D) = \text{defined in Eq. A-4}. \]

\[ \delta_j (z_D) = \text{dimensionless temperature functions of wellbore}. \]

\[ \xi_j (z_D) = \text{dimensionless function (Eq. A-5)}. \]

\[ \rho = \text{density, kg/m}^3 \text{ [lbm/ft}^3]. \]

\[ \rho_f = \text{fluid density in tubing, kg/m}^3 \text{ [lbm/ft}^3]. \]

\[ \sigma_j = \text{dimensional constant (Eq. A-6)}. \]

\[ \phi_j (r_D, z_D, r_D) = \text{dimensionless temperature function of formation}. \]

\[ \omega_j = \text{defined in Eq. A-13}. \]

**Subscripts**

\( D \) = dimensionless

\( j \) = formation layer index \((j=1,2...n)\)

\( 1 \) = in tubing

\( 2 \) = in formation

**Superscripts**

\( - \) = Laplace space

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**References**


**Appendix A—Analytical Solution in Laplace Space**

In terms of the dimensionless variables defined in Eqs. 10 through 14, the problem becomes

\[ \frac{\partial \phi_j}{\partial t_D} + \beta_j (\theta_j - \phi_j | r_D = 1) + \delta_j (z_D) = 0 \quad (A-1) \]

and

\[ \frac{\partial^2 \phi_j}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial \phi_j}{\partial r_D} = \sigma_j \frac{\partial \phi_j}{\partial t_D} \quad (A-2) \]

where \(j=1,2...n\) and

\[ \beta_j = \frac{U_j D / \rho \rho_f c_r r_D v_m}{1} \quad (A-3) \]

\[ \delta_j (z_D) = \begin{cases} \xi_j (z_D) & \text{for liquid flow} \\ \xi_j (z_D) \pm D / (T_{inj} - T_{air}) c_r & \text{for gas flow} \end{cases} \quad (A-4) \]

\[ \xi_j (z_D) = DT_{ij} (z) / (T_{inj} - T_{air}) \quad (A-5) \]

and \( \sigma_j = \rho_f c_r v_m r_D^2 / D k_{hf} \quad (A-6) \)

with initial conditions

\[ \theta_j (z_D=0, r_D=0) = 0 \quad (A-7) \]

and boundary conditions

\[ \theta_1 (z_D=0, r_D=1) = 1 \quad (A-9) \]

\[ \phi_j (r_D=1, z_D=0) = 0 \quad (A-10) \]

and

\[ \lim_{r_D \to \infty} \phi_j (r_D, z_D, r_D) = 0 \quad (A-12) \]
In Eq. A-11,
\[ \omega_j = \frac{U_j r_j}{K_j} \]  
(A-13)
The Laplace transforms of \( \delta_j(z_D, t_D) \) and \( \phi_j(r_D, z_D, t_D) \) are defined as follows:
\[ \tilde{\delta}_j(z_D, s) = \int_0^\infty \delta_j(z_D, t_D) e^{-s \tau^j_D} d\tau^j_D, j = 1, 2, \ldots n \]  
(A-14)
and \( \tilde{\phi}_j(r_D, z_D, s) = \int_0^\infty \phi_j(r_D, z_D, t_D) e^{-s \tau^j_D} d\tau^j_D, j = 1, 2, \ldots n \)  
(A-15)

Application of the Laplace transformation to partial differential Eqs. A-1 and A-2 and boundary conditions in Eqs. A-9 through A-12 with incorporation of the initial conditions in Eqs. A-7 and A-8 yields
\[ \frac{d\tilde{\delta}_j}{d\tau^j_D} + (s + \beta_j) \tilde{\delta}_j - \beta_j \tilde{\phi}_j |_{r_D=1} + \frac{\tilde{\phi}_j}{s} (z_D)/s = 0, \]  
(A-16)
\[ \frac{d^2 \tilde{\phi}_j}{d\tau^j_D^2} + (1/r_D^j) \frac{d\tilde{\phi}_j}{d\tau^j_D} - \sigma_j \tilde{\phi}_j = 0, \]  
(A-17)
\[ \frac{d\tilde{\phi}_j}{d\tau^j_D} |_{r_D=1} = \omega_j (\tilde{\phi}_j |_{r_D=1} - \tilde{\delta}_j), \]  
(A-18)
\[ \tilde{\delta}_j(z_D, 0, s) = 1/s, \]  
(A-19)
and \( \tilde{\delta}_j(z_D, 1-s) = \tilde{\delta}_j(z_D, 1-s) \)  
(A-20)
where \( j = 1, 2, \ldots n \). The solutions of Eqs. A-16 and A-17 in Laplace space, satisfying the boundary condition of Eqs. A-18 through A-20, are
\[ \tilde{\delta}_j(z_D, s) = \frac{\delta_j}{2} y_j(z_D) \]  
(A-21)
and \( \tilde{\phi}_j(r_D, z_D) = \omega_j K_j (\sqrt{\sigma_j s r_D}) \) \( \frac{[\omega_j K_j (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s})]}{[\omega_j K_0 (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s})]} \)  
(A-22)
where the \( y_j(z_D) \) are the particular solutions of Eq. A-16 that depend on the initial temperature profile.
\[ C_1 = (1/s) - Y_1(z_D, 0, s), \]  
(A-23)
\[ C_2 = [\tilde{\delta}_j(z_D, 1-s) - Y_j(z_D, 1-s)] \]  
and \( D_j(s) = \beta_j \omega_j K_0 (\sqrt{\sigma_j s}) \) \( [\omega_j K_0 (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s})] \)  
(A-24)

Heat flux into (or from) the formation in Layer j is defined as
\[ \frac{2\pi r^2}{2} (z_D) = -2\pi r^2 (z_D) \frac{\partial \delta_j}{\partial \tau^j_D} |_{r=r_D} \]  
(A-25)
so that \( q_j(z_D) = \int_{T^j_D} (T_{ej} - T^j_D) (\partial \phi_j / \partial \tau^j_D) |_{r_D=1} \]
\[ = -U_j (T^j_D - T^j_D) (\partial \phi_j / \partial \tau^j_D) |_{r_D=1} \]  
(A-26)

Total cumulative heat transfer is
\[ Q_c(t) = \int_0^t \sum_{j=1}^n \int_0^\tau^j_D 2\pi r^2 q_j(z_D) dz d\tau \]  
(A-28)
For linear vertical-initial-temperature distributions in each layer of the formation, we can obtain the explicit form of the particular solution \( y_j(z_D) \) from Eq. 23. Then, the expressions for heat flux and total cumulative heat transfer in the Laplace-transformed space can be derived with Eqs. 24 and 25, respectively.

**Appendix B—Analytical Solutions in Real Space**

For a linear initial-temperature solution in each layer of the formation, Eq. A-21 can be written as
\[ \tilde{\delta}_j(z_D, s) = A_j(s) + B_j(z_D, s) - \bar{A}_j(z_D, s) \tilde{\delta}_j(z_D, s), \]  
\[ j = 1, 2, \ldots n \]  
\[ z_D - 1 \leq z_D \leq z_D, \]  
(B-1)
where \( A_j(s) = \{ \delta_j / s + \beta_j - D_j(s) \} = (1/s) \tilde{f}_j(s) \)
\[ \times [\tilde{f}_j(s)] = s A_j(s), \]  
(B-2)
\[ B_j(z_D, s) = \exp[-\beta_j(z_D - z_D_j - 1)] \exp[-\beta_j(z_D - z_D_j - 1)] \times \exp[D_j(s)(z_D - z_D_j - 1)] \]  
(B-3)
and \( \bar{\delta}_0(z_D, s) = 1/s \)  
(B-4)
The dimensionless temperature functions in the formation are
\[ \tilde{\delta}_j(z_D, s) = \frac{\omega_j K_j (\sqrt{\sigma_j s})}{\omega_j K_0 (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s})} \]  
(B-5)
where \( \omega_j K_0 (\sqrt{\sigma_j s} r_D) = \omega_j K_0 (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s}) \) \( \omega_j K_0 (\sqrt{\sigma_j s} r_D) = \omega_j K_0 (\sqrt{\sigma_j s}) + \sqrt{\sigma_j s} K_1 (\sqrt{\sigma_j s}) \)  
(B-6)
Because the functions \( A_j(s) \), \( B_j(z_D, s) \), and \( \tilde{\delta}_j(z_D, s) \) have a branch point at the origin, we have to use the inversion theorem for Laplace transformations by evaluating the contour integral.  
The following inversion can be proven after some algebraic operations.
\[ f_j(t_D) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-i\lambda t} f_j(\lambda) d\lambda \]
\[ = \frac{4\lambda}{2\pi \sigma_j} \int_{-\infty}^{+\infty} e^{-i\lambda^2 \sigma_j} D_j(\lambda) d\lambda \]
\[ = 2 \int_{-\infty}^{+\infty} u \sin \left( \frac{2(z_D - z_D_j)}{\pi D_j(u)} \right) d\lambda \]
\[ \times \exp \left[ \frac{1}{2} (z_D - z_D_j) R_j(u) - \frac{u^2}{\sigma_j} \right] \]  
(B-7)
Then, \( A_j(t_D) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-i\lambda t} f_j(\lambda) d\lambda \)
\[ = \frac{4\lambda}{2\pi \sigma_j} \int_{-\infty}^{+\infty} e^{-i\lambda^2 \sigma_j} D_j(\lambda) d\lambda \]
\[ = 2 \int_{-\infty}^{+\infty} u \sin \left( \frac{2(z_D - z_D_j)}{\pi D_j(u)} \right) d\lambda \]
\[ \times \exp \left[ \frac{1}{2} (z_D - z_D_j) R_j(u) - \frac{u^2}{\sigma_j} \right] \]  
(B-8)

Taking the inverse Laplace transform of Eq. B-1 and using Eqs. B-9 and B-10 and the convolution property of the Laplace transform, we have
\[ \tilde{\theta}_j(z_D, t_D) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-i\lambda t} f_j(\lambda, r_D) d\lambda \]
\[ = \frac{2}{2\pi \sigma_j} \int_{-\infty}^{+\infty} \exp[-\beta_j(z_D - z_D_j - 1)] \exp[D_j(s)(z_D - z_D_j - 1)] \]
\[ \times \exp \left[ \frac{1}{2} (z_D - z_D_j) R_j(u) - \frac{u^2}{\sigma_j} \right] \]  
(B-11)
and \( g_j(r_D, t_D) = e^{-1} \left[ \bar{g}_j(r_D, \omega) \right] = \frac{1}{2 \pi i} \int_0^\infty e^{-\tau \lambda} \left[ ar{g}_j(r_D, \lambda e^{-i \tau}) - \bar{g}_j(r_D, \lambda e^{i \tau}) \right] d\lambda = \frac{2}{\pi \sigma_j \beta_j} \int_0^\infty \frac{ue^{-u^2/\sigma_j}}{D^*(u)} \times \{ Y_0(u) \left[ \omega_j J_0(u) + u J_1(u) \right] - J_0(u) \left[ \omega_j Y_0(u) + u Y_1(u) \right] \} du \) ... (B-12)

Then, \( \phi_j(r_D, \omega) = \int_0^{r_D} \theta_j(r_D, r) g_j(r_D, t_D = r) dr \). ... (B-13)

In the above solutions,

\( R_j(u) = J_0(u) \left[ \omega_j J_0(u) + u J_1(u) \right] + Y_0(u) \left[ \omega_j Y_0(u) + u Y_1(u) \right] \) ... (B-14)

and \( D_j^\ast (u) = \{ [\omega_j Y_0(u) + u Y_1(u)]^2 + [\omega_j J_0(u) + u J_1(u)]^2 \} / \beta_j \omega_j \). ... (B-15)

For Layer 1, the solution in Eq. B-11 is simplified to Eq. 26.